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COMPARISON OF KINETIC THEORY ANALYSES OF LINEARIZED HEAT TRANSFER BETWEEN PARALLEL PLATES

P. BASSANINI,* C. CERCIGNANI† and C. D. PAGANI*

Applicazioni e Ricerche Scientifiche, S.p.A., Milano, and Instituto di Scienze Fisiche, University of Milano, Milano

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Abstract—Linearized heat transfer between two parallel plates is considered for inverse Knudsen numbers ranging from 0 to 10. The Bhatnagar, Gross and Krook model is used and transformed into a couple of integral equations for density and temperature. These equations are solved numerically. Besides a variational calculation of the solution is made by introducing simple trial functions in a suitable variational principle. The results obtained for the heat flux through the two methods are compared and found in strict agreement (discrepancy less than 0·5 per cent). For the limiting case of the half-space problem, the temperature jump coefficient is evaluated both by a numerical and a variational procedure. The latter gives a value differing by about 0·5 per cent from the value given by the former procedure and of about 1 per cent from the value previously obtained by Welander.

Comparisons are made with the results of the non-linear analysis of Willis, the linearized four moment solution and Takao's experimental data.

NOMENCLATURE

a, ratio of the temperature-jump coefficient to the mean free path in the half-space problem;

B, nondimensional perturbation of the density at the wall;

c, ratio of the molecular velocity vector to $(2RT)^{\frac{1}{2}}$;

c_x, x-component of c;

c_k, approximation to the source function by the discrete ordinate method;

d, distance between the plates [cm];

f, distribution function [g s^3/cm^6];

f_o, Maxwellian distribution function [g s^3/cm^6];

h, perturbation of the Maxwellian f_o;

l, mean free path [cm];

p, pressure [g/cm s^2];

Q, ratio of the x-component of the heat-flux vector to its free molecular value;

R, gas constant [cm^2/degK s^2];

T, unperturbed temperature [degK];

$T \pm \Delta T$, temperatures of the plates [degK];

T_w, wall temperature [degK];

$T_{m,n}$, transcendental functions defined by equation (2.11);

T_n, transcendental functions defined by equation (2.12);

u, v, variables measuring the ratio of x to $\theta(2RT)^{\frac{1}{2}}$;

x, space coordinate [cm];

$\alpha_{h,k}$, coefficients of the approximating system;

δ, $d/[\theta(2RT)^{\frac{1}{2}}]$;

ε_i, $\varphi_i - \mu_i$ $(i = 1, 2)$;

θ, mean free time [s];

μ, viscosity coefficient [g/cm s];

μ_i, asymptotic value of φ_i $(i = 1, 2)$;

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ρ_i, perturbations of density and temperature with respect to ρ and T respectively ($i = 1, 2$);
ρ, average density [g/cm^3];
φ_i, perturbations of density and temperature in the half-space problem ($i = 1, 2$);
ψ_k, approximation to ψ_i by the discrete ordinate method;
ψ_i, $\rho_i \Delta T/T$ ($i = 1, 2$);
$\alpha, \beta, a, b, c_{ik}$, auxiliary quantities.

All quantities whose dimensions have not been specified above are to be considered as dimensionless.

1. INTRODUCTION

HEAT transfer between parallel plates is a problem which has been frequently considered by people interested in rarefied gas-dynamics [1–7]. It contends to linearized Couette flow the place of the most studied boundary value problem in kinetic theory. In particular we recall that the so called BGK model, introduced by Bhatnagar, Gross and Krook [8] in the kinetic theory of ionized gases, was about contemporarily and independently introduced in rarefied gas dynamics by Welander [9] in order to deal with the problem of plane heat transfer. The problem considered by Welander is a limiting case when one of the plates is pushed to infinity. Considering this problem turns out to be useful to evaluate the temperature-jump coefficient.

After Welander's analysis heat transfer was frequently used as a test-bed for approximated methods, but the linearized heat transfer problem was never solved by using the BGK model with such accuracy as was done, e.g. for linearized plane Poiseuille flow [11] and, more recently, for problems with cylindrical symmetry, as Poiseuille flow in tubes of both circular [12] and annular section [13] and Couette flow between concentric cylinders [14].

We have, it is true, the accurate numerical solutions obtained by Willis [6] and Anderson [15] and this could be considered a sufficient reason to rule out any linearized treatment as superfluous. However, we have various reasons to consider linearized heat transfer between parallel plates as an interesting problem to be solved by the BGK model. Firstly, it seems natural to complete the set of typical linearized plane problems which have been solved by an accurate numerical procedure, and add heat transfer to Couette and Poiseuille flow. Secondly, we note that the linearization condition $\Delta T/T \ll 1$ cannot be violated strongly if $T \pm \Delta T$ are the temperatures of the plates. As a matter of fact, the results obtained by Willis [6] show that a proper choice of the parameters allows us to give a wide range of validity to the results obtained by a linearized treatment. It is not known whether this is a peculiar property of the BGK model or not. In any case, it seems worthwhile to have an accurate numerical solution of the linearized problem for comparison purposes, since the relation between the linearized BGK model and the linearized Boltzmann equation is much clearer than the relation between corresponding non-linear equations. Thirdly, we note that a new method of attacking linearized problems has been recently proposed [16]. This method is based on a variational principle and shows promising features as concerns its applications to complicated problems and models more general than BGK. This variational procedure was previously applied to pure shear flows, such as plane Couette and Poiseuille flows and gave results in an exceptional agreement with the numerical solution, at least as far as such overall quantities as the drag in Couette flow or the flow rate in Poiseuille flow are concerned. Therefore it has seemed worthwhile to apply this technique to a problem of different nature such as the heat-transfer problem. This has been done in order to check that the accuracy of the method is largely independent of the nature of the problem. This, in turn, insures that the technique can be

applied with confidence to more complicated problems, e.g. flows in cylindrical geometry, heat transfer from a sphere, flow past a spherical body, etc.

Finally, it seemed also worthwhile to consider the half-space problem, i.e. the limiting case considered by Welander, and set up an accurate numerical technique for its solution. This not only allows a check on Welander's result for the jump coefficient, but also gives the opportunity of studying techniques which can be extended to more difficult and important problems implying boundary conditions at infinity (external flows).

In Section 2 the general formulation of the problem is given, in Section 3 the limiting case of the half-space is considered. In Section 4 we describe the numerical techniques, in Section 5 the variational procedure. Finally, in Section 6 the results are given and various comparisons are made.

2. GENERAL FORMULATION OF THE PROBLEM

Axes are taken with the origin halfway between the parallel plates, which become $x = \pm d/2$. The temperatures of the plates are $T_1 = T + \Delta T$ (at $x = -d/2$) and $T_2 = T - \Delta T$ (at $x = d/2$). The problem is linearized by postulating that $\Delta T/T \ll 1$. The molecular boundary conditions at each wall are fixed by assuming that the molecules emitted have a Maxwellian distribution function characterized by the wall temperature and zero mass velocity. Further conditions are that the net mass flow at the walls be zero, and that the total number of molecules between the walls be constant.

Using the linearized version of the BGK model, the transport equation becomes

$$c_x \frac{\partial h}{\partial x} + \frac{h}{\theta} = \frac{1}{\theta}\left\{\pi^{-\frac{3}{2}} \iiint \exp\left[-c_1^2\right] h(x, c_1)\, dc_1 + \tfrac{2}{3}\pi^{-\frac{3}{2}}(c^2 - \tfrac{3}{2}) \iiint (c_1^2 - \tfrac{3}{2}) \exp\left[-c_1^2\right] h(x, c_1)\, dc_1\right\} \tag{2.1}$$

where h is the perturbation of a Maxwellian characterized by the average density and temperature ρ and T

$$f_o = \rho(2\pi RT)^{-\frac{3}{2}} \exp\left[-c^2\right]; \qquad f = f_o(1 + h). \tag{2.2}$$

The molecular velocity c is measured in $(2RT)^{\frac{1}{2}}$ units, R being the gas constant. θ is the mean free time related to viscosity and pressure by

$$\theta = \frac{3}{2}\frac{\mu}{p}. \tag{2.3}$$

x is the space coordinate in the direction orthogonal to the plates.

The boundary conditions of diffuse reflection can be written as follows

$$h\left(c_x, -\frac{d}{2}\operatorname{sgn} c_x\right) = \frac{\Delta T}{T}(c^2 + B)\operatorname{sgn} c_x \tag{2.4}$$

where B is a constant related to the density of the re-emitted molecules. B is fixed by the mass conservation at the walls, which implies

$$\iiint \exp\left[-c^2\right] c_x\, h(x, c_x)\, dc = 0. \tag{2.5}$$

This relation holds not only at the walls ($x = \pm d/2$), but for any x, since its left-hand side is constant as a consequence of equation (2.1).

Let us now introduce the following quantities

$$\rho_1(x) = \pi^{-\frac{3}{2}} \iiint \exp[-c_1^2]\, h(x, c_1)\, \mathrm{d}c_1 \tag{2.6}$$

$$\rho_2(x) = \tfrac{2}{3}\pi^{-\frac{3}{2}} \iiint (c_1^2 - \tfrac{3}{2}) \exp[-c_1^2]\, h(x, c_1)\, \mathrm{d}c_1. \tag{2.7}$$

ρ_1 and ρ_2 are suitable expressions for nondimensional density and temperature respectively. Introducing equations (2.6) and (2.7) into equation (2.1) allows this equation to be integrated to give a couple of "pure" integral equations, incorporating also the boundary conditions:

$$\psi_1(u) = -\pi^{-\frac{3}{2}}\{[T_{0,1}(\delta/2 - u) - T_{0,1}(\delta/2 + u)] + (\tfrac{3}{2} + B)[T_{0,0}(\delta/2 - u) - T_{0,0}(\delta/2 + u)] + - \int_{-\delta/2}^{+\delta/2} [T_{-1,0}(|u - v|)\psi_1(v) + T_{-1,1}(|u - v|)\psi_2(v)]\, \mathrm{d}v\} \tag{2.8}$$

$$\psi_2(u) = -\tfrac{2}{3}\pi^{-\frac{3}{2}}\{[T_{0,2}(\delta/2 - u) - T_{0,2}(\delta/2 + u)] + (\tfrac{3}{2} + B)[T_{0,1}(\delta/2 - u) - T_{0,1}(\delta/2 + u)] + - \int_{-\delta/2}^{+\delta/2} [T_{-1,1}(|u - v|)\psi_1(v) + T_{-1,2}(|u - v|)\psi_2(v)]\mathrm{d}v\}. \tag{2.9}$$

Here we have put

$$u = \frac{x}{\theta}; \qquad \delta = \frac{d}{\theta}; \qquad \rho_i(x) = \psi_i(u)\, \Delta T/T \ (i = 1, 2); \tag{2.10}$$

while the functions $T_{m,n}$ are defined as follows

$$T_{m,n}(x) = \int_{-\infty}^{+\infty}\int \mathrm{d}a\, \mathrm{d}b \int_0^{\infty} \mathrm{d}c\, c^m (a^2 + b^2 + c^2 - \tfrac{3}{2})^n \exp[-a^2 - b^2 - c^2 - (x/c)] \tag{2.11}$$

and can be easily reduced to the functions $T_n(x)$, which are defined by the quadrature

$$T_n(x) = \int_0^{\infty} c^n \exp[-c^2 - (x/c)]\, \mathrm{d}c \tag{2.12}$$

and are familiar to people dealing with the BGK model [9, 17, 18]. In particular we quote the following useful relations

$$\begin{aligned} T_{m,0}(x) &= \pi T_m(x) \\ T_{m,1}(x) &= \pi T_{m+2}(x) - (\pi/2)\, T_m(x) \\ T_{m,2}(x) &= \pi T_{m+4}(x) - \pi T_{m+2}(x) + \tfrac{5}{4}\pi T_m(x). \end{aligned} \tag{2.13}$$

Equation (2.5) now becomes

$$T_{1,1}(\delta/2 + u) + T_{1,1}(\delta/2 - u) + (B + \tfrac{3}{2})[T_{1,0}(\delta/2 + u) + T_{1,0}(\delta/2 - u)] + \int_{-\delta/2}^{+\delta/2} \operatorname{sgn}(u - v)[T_{0,0}(|u - v|)\psi_1(v) + T_{0,1}(|u - v|)\psi_2(v)]\, \mathrm{d}v = 0. \tag{2.14}$$

The ratio of the component of the heat-transfer vector along the x-axis to the corresponding value in free-molecular conditions is

$$Q = \frac{q_x}{q_{x\mathrm{fm}}} = \frac{1}{2\pi}\frac{T}{\Delta T}\iiint c_x c^2 \exp[-c^2]\, h(x, c_x)\, \mathrm{d}c = \frac{1}{2\pi}[T_{1,2}(\delta/2 + u)$$

$$+ (3 + B)\, T_{1,1}(\delta/2 + u) + \tfrac{3}{2}(\tfrac{3}{2} + B)\, T_{1,0}(\delta/2 + u) + T_{1,2}(\delta/2 - u)$$
$$+ (3 + B)\, T_{1,1}(\delta/2 - u) + \tfrac{3}{2}(\tfrac{3}{2} + B)\, T_{1,0}(\delta/2 - u)]$$
$$+ \frac{1}{2\pi} \int_{-\delta/2}^{+\delta/2} \operatorname{sgn}(u - v)\{[T_{0,1}(|u - v|) + \tfrac{3}{2}\, T_{0,0}(|u - v|)]\, \psi_1(v)$$
$$+ [T_{0,2}(|u - v|) + \tfrac{3}{2}\, T_{0,1}(|u - v|)]\, \psi_2(v)\}\, dv. \tag{2.15}$$

Although Q is formally depending on u, it is actually a constant throughout the gap between the plates, thanks to the energy conservation equation. The same thing can be said about the left side of equation (2.14), thanks to the mass conservation equation.

3. THE TEMPERATURE JUMP PROBLEM

If $\delta \gg 1$ the problem of heat conduction can be analysed by using the conventional continuum equations, but replacing the no-jump condition for temperature with the condition

$$T - T_w = al \frac{\partial T}{\partial n} \tag{3.1}$$

where T_w is the wall temperature, T the temperature and $\partial T/\partial n$ the normal derivative of the temperature at the wall according to the Navier–Stokes level of description, a a numerical constant, l the mean free path related to our mean free time by

$$\theta = 3\pi^{-\frac{1}{2}} l; \qquad [(2RT)^{\frac{1}{2}} = 1]. \tag{3.2}$$

From the microscopic point of view $(\partial T/\partial n)_w$ is the temperature slope at many (in practice, 10 or less) mean free paths from the wall. It is now of interest to determine the constant a from a kinetic description. This was done for the BGK equation by Welander [9] who found $a = (1{\cdot}173)\, \frac{15}{8}$, $\frac{15}{8}$ being the classical value obtained by Maxwell through an approximated reasoning.

It seems worthwhile to check Welander's results, since his numerical result of the corresponding constant in the slip coefficient turned out to be in error of about 10 per cent [10, 19].

Besides the temperature jump problem offers the possibility of testing numerical techniques which are thought to be useful in more complicated external flows, e.g. the flow past a body.

The integral equations governing the density and temperature for this problem can be easily obtained either directly or from equations (2.8), (2.9) after factoring $1/\delta$ out from ψ_i and going to the limit $\delta \to \infty$. The result is

$$\varphi_1(u) = \pi^{-\frac{1}{2}} \{T_{1,1}(u) - T_{1,0}(u) + \int_0^\infty T_{-1,0}(|u - v|)\, \varphi_1(v)\, dv + \int_0^\infty T_{1,1}(|u - v|)\, \varphi_2(v)\, dv\} \tag{3.3}$$

$$\varphi_2(u) = \tfrac{2}{3}\pi^{-\frac{1}{2}} \{T_{1,2}(u) - T_{1,1}(u) + \int_0^\infty T_{-1,1}(|u - v|)\, \varphi_1(v)\, dv + \int_0^\infty T_{-1,2}(|u - v|)\, \varphi_2(v)\, dv\} \tag{3.4}$$

where the density $\rho(x)$ and the temperature $T(x)$ are related to $\varphi_i(u)$ $(i = 1, 2)$ by

$$\rho(x) = \rho + \frac{\rho\theta}{T}\left(\frac{\partial T}{\partial n}\right) \varphi_1(u) \tag{3.5}$$

$$T(x) = T + \theta\left(\frac{\partial T}{\partial n}\right) \varphi_2(u). \tag{3.6}$$

It is also useful to consider another couple of functions $\varepsilon_i(u)$ $(i = 1, 2)$ related to φ_i by

$$\varphi_i(u) = \mu_i + \varepsilon_i(u); \qquad \mu_i = \lim_{u \to \infty} \varphi_i(u). \tag{3.7}$$

In terms of these new unknowns, equations (3.3) and (3.4) become

$$\varepsilon_1(u) = \pi^{-\frac{1}{2}} \{ -\mu_1 T_{0,0}(u) - \mu_2 T_{0,1}(u) + T_{1,1}(u) - T_{1,0}(u) + \int_0^\infty T_{-1,0}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty T_{-1,1}(|u - v|)\, \varepsilon_2(v)\, dv \} \tag{3.8}$$

$$\varepsilon_2(u) = \tfrac{2}{3} \pi^{-\frac{1}{2}} \{ -\mu_1 T_{0,1}(u) - \mu_2 T_{0,2}(u) + T_{1,2}(u) - T_{1,1}(u) + \int_0^\infty T_{-1,1}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty T_{-1,2}(|u - v|)\, \varepsilon_2(v)\, dv \}. \tag{3.9}$$

It is also useful to write down the two systems of equations which are obtained from equations (3.8) and (3.9) by integrating once and twice respectively

$$\left.\begin{aligned} &\int_0^\infty \operatorname{sgn}(u - v)\, T_{0,0}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty \operatorname{sgn}(u - v)\, T_{0,1}(|u - v|)\, \varepsilon_2(v)\, dv \\ &\qquad = \mu_1 T_{1,0}(u) + \mu_2 T_{1,1}(u) - T_{2,1}(u) + T_{2,0}(u) \\ &\int_0^\infty \operatorname{sgn}(u - v)\, T_{0,1}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty \operatorname{sgn}(u - v)\, T_{0,2}(|u - v|)\, \varepsilon_2(v)\, dv \\ &\qquad = \mu_1 T_{1,1}(u) + \mu_2 T_{1,2}(u) - T_{2,2}(u) + T_{2,1}(u) \end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned} &\int_0^\infty T_{1,0}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty T_{1,1}(|u - v|)\, \varepsilon_2(v)\, dv \\ &\qquad = \mu_1 T_{2,0}(u) + \mu_2 T_{2,1}(u) - T_{3,1}(u) + T_{3,0}(u) \\ &\int_0^\infty T_{1,1}(|u - v|)\, \varepsilon_1(v)\, dv + \int_0^\infty T_{1,2}(|u - v|)\, \varepsilon_2(v)\, dv \\ &\qquad = \mu_1 T_{2,1}(u) + \mu_2 T_{2,2}(u) - T_{3,2}(u) + T_{3,1}(u) \end{aligned}\right\} \tag{3.11}$$

4. NUMERICAL TECHNIQUES OF SOLUTION

As in the previous case of plane Poiseuille flow [11], we have solved equations (2.8) and (2.9) by numerical techniques. Also in this problem the most obvious way of differencing these equations cannot be used because of the singularity of the kernel for vanishing argument. However, since this singularity of the kernel is integrable, we considered the following system

$$\sum_{k=-n}^{n} \alpha_{hk}\, \psi_k = c_h \qquad (h = 1, \ldots, n) \tag{4.1}$$

where $\psi_0 = (\frac{3}{2} + B)$, ψ_k $(k = -n, \ldots, -1)$ approximates $\psi_1(u)$ for $u = u_{|k|}$, ψ_k $(k = 1, \ldots, n)$ approximates $\psi_2(u)$ for $u = u_k$, and the α_{hk} and c_h are given by formulas simple but too long to be reported here.

A differencing method similar to method (a) of reference [11] was used. The resulting difference equations have the following simple interpretation: the functions $\psi_i(u)$ $(i = 1, 2)$ are approximated by a stepwise function and the constant value on every interval is interpreted as the value in the midpoint. Besides B is assumed to be fixed by equation (2.14) written in a discrete fashion for $u = -\delta/2$.

We note also that in many cases we used, as in reference [11], a differencing procedure with equal steps. However, for small δ a procedure based on taking intervals centered at the zeroes of the Legendre polynomials was found more accurate and accordingly preferred.

Similar procedures were used to write down the discrete analog of equation (2.15) giving the non-dimensional heat transfer.

In the case of the half-space problem we proceeded as follows. We wrote equations (3.8) in a discrete fashion by using a mesh of variable stepsize. In order to determine μ_i $(i = 1, 2)$ we differenced also the formulas obtained by equations (3.10) by putting $u = 0$. Then the system of resulting equations was solved numerically. The meshsteps were taken very small near the wall, but their size increased exponentially with the distance from the wall. This choice was dictated by the circumstance that the ε_i $(i = 1, 2)$ are very small some mean free path far from the wall (practically zero for $u = 10$), but are very rapidly varying near the wall. As a test of good approximation we evaluated with an analogous method the solution of the half-space shear flow problem and found a good agreement with the available solutions [10, 20, 21].

The application of this method appears rather interesting, since we hope to apply it to external flow problems, the general procedure being that of subtracting the asymptotic behavior (in our case $\varphi_i \sim \mu_i$) and then solve for the remaining part of the unknowns with a fine mesh near the wall.

5. THE VARIATIONAL METHOD

A general variational procedure applying to linearized kinetic models has been introduced in reference [16]. With reference to this paper we shall omit any details and consider the following functional

$$J(\tilde{\boldsymbol{\psi}}, \tilde{B}) = (\tilde{\boldsymbol{\psi}}, \pi^{\frac{1}{2}}\, \tilde{\boldsymbol{\psi}} - \mathfrak{S}\, \tilde{\boldsymbol{\psi}} - 2\boldsymbol{S}) + \alpha \tilde{B}^2 + 2\beta \tilde{B} \tag{5.1}$$

where $\tilde{\boldsymbol{\psi}}(u)$ is a two-dimensional vector

$$\tilde{\boldsymbol{\psi}}(u) = \begin{vmatrix} \tilde{\psi}_1(u) \\ (\sqrt{\tfrac{3}{2}})\, \tilde{\psi}_2(u) \end{vmatrix} \tag{5.2}$$

and $\mathfrak{S}$ a two-by-two matrix operator such that

$$\mathfrak{S}\tilde{\boldsymbol{\psi}} = \int_{-\delta/2}^{+\delta/2} \begin{vmatrix} T_{-1,0}(|u - v|) & (\sqrt{\tfrac{2}{3}})\; T_{-1,1}(|u - v|) \\ (\sqrt{\tfrac{2}{3}})\; T_{-1,1}(|u - v|) & \tfrac{2}{3}\; T_{-1,2}(|u - v|) \end{vmatrix} \cdot \begin{vmatrix} \tilde{\psi}_1(u) \\ \tilde{\psi}_2(u) \end{vmatrix} \mathrm{d}v. \tag{5.3}$$

Besides

$$\alpha = 2[T_{1,0}(\delta) + T_{1,0}(0)] \tag{5.4}$$

$$\beta = 2[T_{1,1}(\delta) + \tfrac{3}{2}\, T_{1,0}(\delta) + T_{1,1}(0) + \tfrac{3}{2}\, T_{1,0}(0)] \tag{5.5}$$

$$\boldsymbol{S}(u) = -\begin{vmatrix} [T_{0,1}(\delta/2 - u) - T_{0,1}(\delta/2 + u)] + (\tfrac{3}{2} + \tilde{B})[T_{0,0}(\delta/2 - u) - T_{0,0}(\delta/2 + u)] \\ (\sqrt{\tfrac{2}{3}})[T_{0,2}(\delta/2 - u) - T_{0,2}(\delta/2 + u)] + (\sqrt{\tfrac{2}{3}})(\tfrac{3}{2} + \tilde{B})[T_{0,1}(\delta/2 - u) \\ \qquad + T_{0,1}(\delta/2 + u)] \end{vmatrix}. \tag{5.6}$$

The functional (5.1) is such that, when varying $\boldsymbol{\psi}$ and B, it attains its minimum value when $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$ and $\tilde{B} = B$, i.e. $\tilde{\boldsymbol{\psi}}$ and $\tilde{B}$ satisfy equations (2.8) and (2.9) together with equation (2.14) (actually

the average between the relations obtained from this equation by putting $u = \pm\delta/2$). The minimum value attained by $J(\tilde{\psi}, \tilde{B})$ is

$$\begin{aligned} J(\psi, B) &= 2B[T_{1,1}(\delta) + \tfrac{3}{2}T_{1,0}(\delta) + T_{1,1}(0) + \tfrac{3}{2}T_{1,0}(\delta)] \\ &+ \int_{-\delta/2}^{+\delta/2} \{[T_{0,1}(\delta/2 - u) - T_{0,1}(\delta/2 + u)] + \tfrac{3}{2}[T_{0,0}(\delta/2 - u) - T_{0,0}(\delta/2 + u)]\}\, \psi_1(u)\, du \\ &+ \int_{-\delta/2}^{+\delta/2} \{[T_{0,2}(\delta/2 - u) - T_{0,2}(\delta/2 + u)] + \tfrac{3}{2}[T_{0,1}(\delta/2 - u) - T_{0,1}(\delta/2 + u)]\}\, \psi_2(u)\, du. \end{aligned} \tag{5.7}$$

Now if we evaluate equation (2.15) at $u = \pm\delta/2$ we find, by taking the arithmetical mean

$$Q = \frac{1}{4\pi} J(\psi, B) + \frac{1}{2\pi}\left[T_{1,2}(\delta) + 3T_{1,1}(\delta) + \frac{9}{4}T_{1,0}(\delta) + T_{1,2}(0) + 3T_{1,1}(0) + \frac{9}{4}T_{1,0}(0)\right]. \tag{5.8}$$

Therefore we achieve the result that the value of the heat flux is strictly related to the minimum value attained by the functional J.

If we use the following trial function for $\tilde{\psi}$

$$\tilde{\psi}(u) = \begin{vmatrix} au \\ (\sqrt{\tfrac{3}{2}})\, bu \end{vmatrix} \tag{5.9}$$

where a and b are two available constants, we find

$$J(\tilde{\psi}, \tilde{B}) = c_{11}a^2 + c_{22}b^2 + c_{33}B^2 + 2(c_{12}ab + c_{13}aB + c_{23}bB) - 2(c_1 a + c_2 b + c_3 B) \tag{5.10}$$

where

$$\left.\begin{aligned} c_{11} &= \frac{\delta^2}{4} - 1 + \delta T_0(\delta) + \left(\frac{\delta^2}{2} + 2\right) T_1(\delta) + 2\delta T_2(\delta) \\ c_{22} &= \frac{9}{16}\delta^2 - \frac{21}{4} + \frac{\delta}{4}(3\delta^2 + 21)\, T_0(\delta) + \frac{1}{2}\left(\frac{33}{4}\delta^2 + 21\right) T_1(\delta) + \frac{\delta}{2}\left(\frac{\delta^2}{2} + 21\right) T_2(\delta) \\ c_{33} &= 1 + 2T_1(\delta) \\ c_{12} &= \frac{\delta^2}{8} - \frac{3}{2} + \frac{\delta}{2}\left(\frac{\delta^2}{2} + 3\right) T_0(\delta) + \left(\frac{5}{4}\delta^2 + 3\right) T_1(\delta) + 3\delta T_2(\delta) \\ c_{13} &= \frac{\delta}{2} - \frac{\pi^{\frac{1}{2}}}{2} + \delta T_1(\delta) + 2T_2(\delta) \\ c_{23} &= \frac{\delta}{4} - \frac{\pi^{\frac{1}{2}}}{2} + \frac{\delta^2}{2} T_0(\delta) + \frac{3}{2}\delta T_1(\delta) + 2T_2(\delta) \\ c_1 &= -\delta + \frac{5}{4}\pi^{\frac{1}{2}} - \frac{\delta^2}{2} T_0(\delta) - 5T_2(\delta) - 3\delta T_1(\delta) \\ c_2 &= -\frac{3}{2}\delta + \frac{5}{2}\pi^{\frac{1}{2}} - \frac{7}{4}\delta^2 T_0(\delta) - 7\delta T_1(\delta) - \left(\frac{\delta^2}{2} + 10\right) T_2(\delta) \\ c_3 &= -2 - \delta T_0(\delta) - 4T_1(\delta). \end{aligned}\right\} \tag{5.11}$$

Here the two-subscript T's have been reduced to one-subscript T's by means of equations (2.13).

Minimizing $J(\tilde{\psi}, \tilde{B})$ with respect to a, b, B we find a system of linear equations which can be easily solved. The values of min $J(\tilde{\psi}, \tilde{B})$ corresponding to different δ are then given by

$$\min J(\tilde{\psi}, \tilde{B}) = -c_1 a - c_2 b - c_3 B \tag{5.12}$$

and are related to the heat flux by equation (5.8). The results will be given in the following section.

An analogous procedure can be applied to the temperature jump problem. Here we consider the functional*

$$J(\tilde{\varphi}) = (\tilde{\varphi}, \mathfrak{S}\tilde{\varphi} - \pi^{\frac{1}{2}}\tilde{\varphi} + 2\boldsymbol{S}) \tag{5.13}$$

where

$$\tilde{\varphi} = \begin{vmatrix} \tilde{\varphi}_1(u) \\ \sqrt{(\frac{3}{2})}\,\tilde{\varphi}_2(u) \end{vmatrix} \tag{5.14}$$

$$\boldsymbol{S}(u) = \begin{vmatrix} T_{1,1}(u) - T_{1,0}(u) \\ \sqrt{(\frac{2}{3})}\,[T_{1,2}(u) - T_{1,1}(u)] \end{vmatrix} \tag{5.15}$$

We also find from equations (3.11), by putting $u = 0$, $\varepsilon_i = \varphi_i - \mu_i$ $(i = 1, 2)$

$$(\tilde{\varphi}, \boldsymbol{S}) = \frac{5}{4}\pi^{\frac{1}{2}}\mu_2 - \frac{13}{8}\pi \tag{5.16}$$

i.e.

$$\mu_2 = \tfrac{13}{10}\pi^{-\frac{1}{2}} + \tfrac{4}{5}\pi^{-\frac{3}{2}}(\tilde{\varphi}, \boldsymbol{S}) = \tfrac{13}{10}\pi^{-\frac{1}{2}} + \tfrac{4}{3}\pi^{-\frac{1}{2}} \max J(\tilde{\varphi}). \tag{5.17}$$

Since equations (3.1), (3.2), (3.6) and (3.7) give

$$\mu_2 = \frac{\pi^{\frac{1}{2}}}{3}a, \tag{5.18}$$

equation (5.17) relates the value of the temperature-jump coefficient to the maximum attained by J.

Now if we take a trial vector $\tilde{\varphi}$ given by the constant vector

$$\boldsymbol{c} = \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} \tag{5.19}$$

we find

$$J(\boldsymbol{c}) = -\frac{\pi}{2}c_1^2 - \frac{\pi}{6^{\frac{1}{2}}}c_1c_2 - \frac{3}{4}\pi c_2^2 + \frac{5}{2}\pi\left(\frac{\pi}{6}\right)^{\frac{1}{2}}c_2. \tag{5.20}$$

Therefore

$$\max J(\boldsymbol{c}) = \frac{25}{64}\pi^2. \tag{5.21}$$

---

* The range of integrations of the integrals appearing in equation (5.3) and in the inner product is now obviously changed from $[-(\delta/2), +(\delta/2)]$ to $(0, \infty)$.

It follows that according to the present simple approximation to φ we find

$$\mu_2 = \frac{5}{8}\pi^{\frac{1}{2}}\left(\frac{1}{2} + \frac{52}{25\pi}\right) \tag{5.22}$$

and consequently

$$a = \left[\frac{1}{2} + \frac{52}{25\pi}\right]\frac{15}{8}. \tag{5.23}$$

6. RESULTS

The heat flux resulting from the numerical and variational methods is given in Table 1. It will be seen that the disagreement is very small (less than 0·5 per cent for $\delta \leqslant 10$). This circumstance constitutes a good test for the accuracy of both methods.

Table 1. *Heat flux vs. inverse Knudsen number*

δ	Numerical solution	Variational method
0·01	0·9925	0·9925
0·1	0·9352	0·9352
0·5	0·7682	0·7683
1·0	0·6405	0·6409
1·25	0·5933	0·5939
1·5	0·5532	0·5539
1·75	0·5194	0·5194
2·0	0·4893	0·4894
2·5	0·4390	0·4391
3·0	0·3985	0·3986
4·0	0·3370	0·3370
5·0	0·2923	0·2922
7·0	0·2314	0·2311
10·0	0·1767	0·1760

In Fig. 1 the resulting curve is compared with the four moment result

$$Q = [1 + 4\delta/5\pi^{\frac{1}{2}}]^{-1}. \tag{6.1}$$

It is also to be noted that for $\delta \leqslant 1{\cdot}5$ the numerical results show an almost complete coincidence with the results obtained by Wang Chang and Uhlenbeck [1] for Maxwell molecules.

It appears interesting to consider a comparison of the linearized results with the results of Willis's non-linear treatment [6]. To this end it is useful to introduce the rarefaction parameter r suggested by Willis. r is simply equal to $8d/5\pi^{\frac{1}{2}}\theta^*$ provided that θ is evaluated for a temperature T' given by

$$T' = \frac{1}{4}\left[(T + \Delta T)^{\frac{1}{2}} + (T - \Delta T)^{\frac{1}{2}}\right]^2 \left[1 - \left(\frac{\Delta T}{T}\right)^2\right]^{-1}. \tag{6.2}$$

* We note that equations (31) and (32) of reference [6] are in contrast and equation (31) should be regarded as correct.

The comparison is shown in Fig. 2. As was already noticed by Willis the correlation between linearized and non-linearized results is extremely good with the adopted definition of r.

A comparison with Takao's experimental data was also made. Although Takao considered heat transfer in air and our results apply to a monoatomic gas, the comparison is possible provided that we are not very near to the continuum region. As a matter of fact, if we plot the ratio of the heat transfer to its free molecule value we take away the dependence on the nature of the gas in the free molecular limit. The disadvantage of this way of plotting is that the experimental errors and the inaccuracy of deducing data from Takao's plot are magnified in the free molecular limit.

In any case, Fig. 3 shows the comparison for δ ranging from 0·2 to 10.

While the variational approach, by its own nature, cannot give accurate temperature and density profiles, but only a linear fit of them,

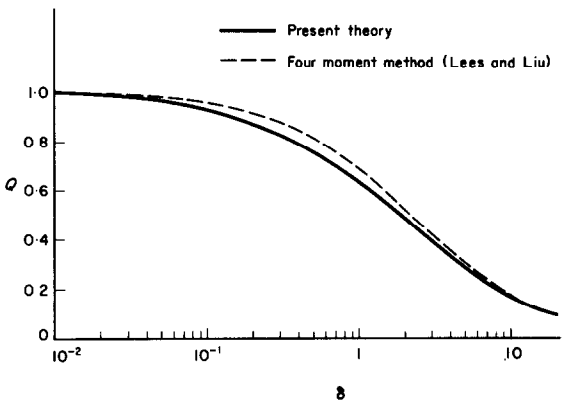

FIG. 1. Comparison of our results with linearized four-moment solution by Lees and Liu's half-range method.

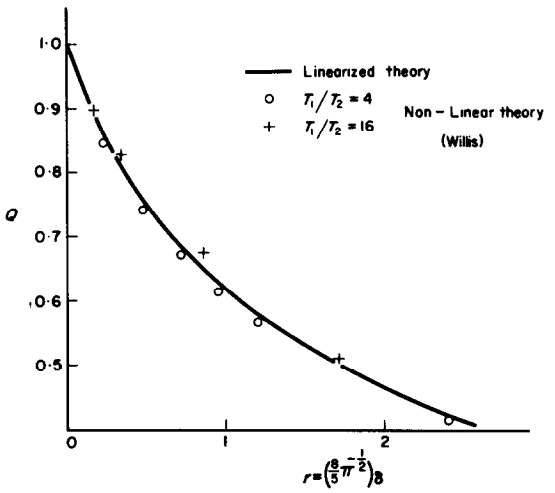

FIG. 2. Comparison of our results with Willis's solution of the nonlinear BGK model. Willis's abscissae are measured in terms of a mean free path corresponding to the average temperature of equation (6.2).

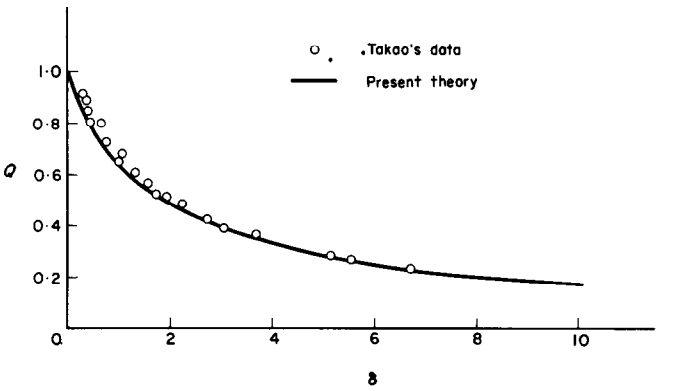

FIG. 3. Comparison with Takao's experimental data.

the numerical solution gives also an accurate representation of the local values of density and temperature. Typical examples of such profiles are given in Figs. 4 and 5.

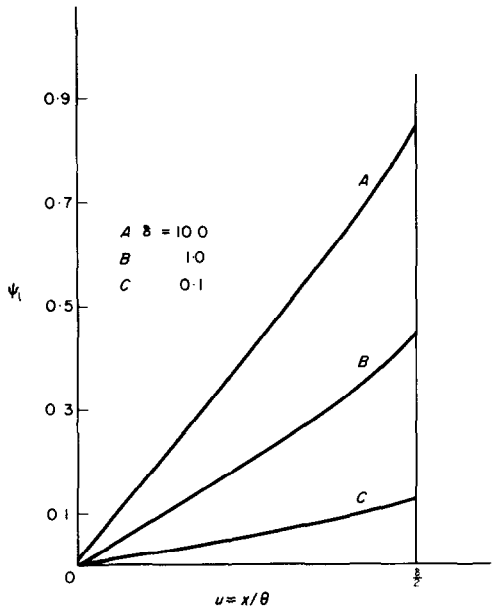

FIG. 4. Profiles of the density perturbation for typical values of the inverse Knudsen number δ.

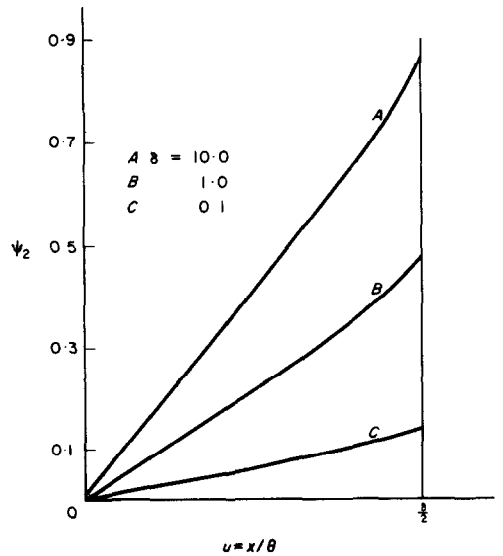

FIG. 5. Profiles of the temperature perturbation for typical values of the inverse Knudsen number δ.

Concerning the half-space problem we evaluated the temperature and density profiles by the finite difference method; the deviations from the linear asymptotic behaviour are shown in Fig. 6.

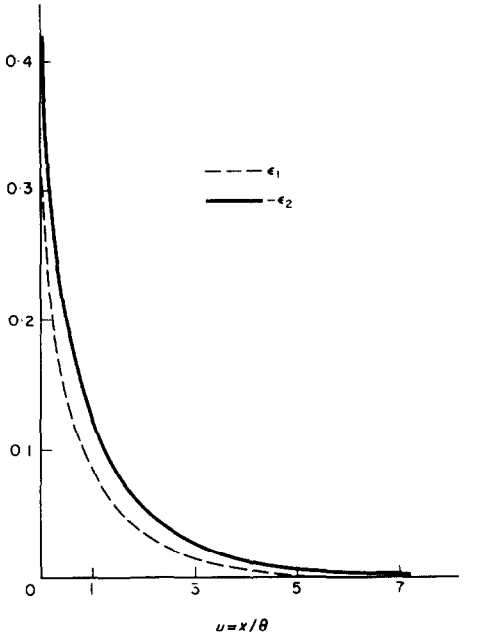

FIG. 6. Profiles of the density and temperature perturbations in the half-space problem.

The same calculation gave the following values for μ_i $(i = 1, 2)$:

$$\left.\begin{aligned} \mu_1 &= -0{\cdot}7319 \\ \mu_2 &= 1{\cdot}2941. \end{aligned}\right\} \quad (6.3)$$

Therefore the temperature-jump coefficient turns out to be:

$$a = 3\pi^{-1/2}(1{\cdot}2941) = \tfrac{15}{8}(1{\cdot}1682) \quad (6.4)$$

The value obtained by the variational method is by equation (5.23):

$$a = \tfrac{15}{8}(1{\cdot}1621). \quad (6.5)$$

The discrepancy is less than 0·53 per cent. The value 1·173 obtained by Welander [9] differs from equation (6.4) by 0·43 per cent and from equation (6.5) by 0·86 per cent.

7. CONCLUDING REMARKS

A typical linearized problem of kinetic theory

has been solved by two different methods, based respectively on a differencing procedure and a variational approach. The agreement between these results is good.

Since the BGK model has been used in place of the true linearized Boltzmann equation, it is pertinent to ask whether the change of the collision term affects significantly the results. Concerning this point, we can quote the result that the most striking inaccuracy of the BGK model, i.e. the fact that the Prandtl number is equal to 1 instead of $\frac{2}{3}$, has not very marked influence on the results, although in the transition regime normal stresses are present, which are certainly influenced by the value of the Prandtl number. To be precise we can say* that if a more sophisticated model with correct Prandtl number is used, as proposed in reference [22], corrections to heat flux in the transition regime are less than 0·5 per cent.

Therefore, the results for the BGK model should be quite accurate also for Maxwell molecules, unless higher order moments have an improbable influence on the results.

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* Kind communication by Dr. Gino Tironi.

Résumé—On considère le transport de chaleur linéarisé entre deux plaques parallèles dans la gamme des inverses du nombre de Knudsen allant de 0 à 10. Le modèle de Bhatnagar, Gross et Krook est employé et transformé en un couple d'équations intégrales pour la densité et la température. Ces équations sont résolues numériquement. De plus, on fait un calcul variationnel de la solution en introduisant des fonctions d'essai simples avec un principe variationnel convenable. Les résultats obtenus pour le flux de chaleur à

l'aide des deux méthodes sont comparés et l'on a constaté leur accord étroit (différence inférieure à 0,5 pour cent). Dans le cas limite du problème du demi-espace, le coefficient de saut de température est évalué à la fois numériquement et par un procédé variationnel. Ce dernier donne une valeur différant d'environ 0,5 pour cent de la valeur donnée par le premier procédé et d'environ 1 pour cent de la valeur obtenue auparavant par Welander.

On a comparé avec les résultats de l'analyse non-linéaire de Willis, la solution linéarisée des quatre moments et les résultats expérimentaux de Takao.

Zusammenfassung—Der linearisierte Wärmeübergang zwischen zwei parallelen Platten wird behandelt für Kehrwerte der Knudsenzahl von 0 bis 10. Das Modell von Bhatnagar, Gross und Krook wird benützt und transformiert in eine Reihe von Integralgleichungen für die Dichte und Temperatur. Diese Gleichungen werden numerisch gelöst. Daneben ist eine Variationsrechnung für die Lösung durchgeführt durch Einführung einfacher Versuchsfunktionen in ein geeignetes Variationsschema. Die nach beiden Methoden erhaltenen Ergebnisse für den Wärmestrom werden verglichen und es ergibt sich strenge Übereinstimmung (Abweichung weniger als 0,5 Prozent). Für den Grenzfall des Halbraumproblems ist der Koeffizient für den Temperatursprung sowohl nach einem numerischen Verfahren als auch nach der Variationsmethode berechnet. Letztere liefert einen Wert, der etwa 0,5 Prozent abweicht von jenem, der nach der erstgenannten Methode ermittelt wird und etwa 1 % Abweichung zu dem kürzlich von Welander erhaltenen Wert aufweist.

Vergleiche wurden durchgeführt mit den Resultaten der nichtlinearen Analyse von Willis, der linearisierten Lösung mit vier Momenten und Takaos experimentellen Daten.

Аннотация—Рассматривается линеаризированная задача теплообмена между двумя параллельными пластинами для обратных значений чисел Кнудсена в диапазоне от 0 до 10. Используется модель Бхатнагара, Гросса и Крука, которая преобразовывается в систему двух интегральных уравнений для плотности и температуры. Эти уравнения решены численно. Кроме того, выполнен вариационный расчет решения путем введения простых пробных функций в соответствии с вариационным принципом. Проведено сравнение результатов, полученных для теплового потока с помощью этих двух методов, и найдено их хорошее соответствие (с отклонением не более 0,5%). Для предельного случая задачи в полупространстве коэффициент скачка температур определяется как численным, так и вариационным методом. Вариационный метод дает значение, отличающееся на ~0,5% от значения, полученного численным методом, и на ~1% — от значения Уэландера.

Проведено сравнение полученных результатов с результатами нелинейного анализа Уиллиса, линеаризированным решением четырех моментов и экспериментальными данными Такао.